View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

JOURNAL OF APPROXIMATION THEORY 13, 17–23 (1975)

Nonuniqueness of Simultaneous Approximation by Algebraic Polynomials

R. A. LORENTZ

Gesellschaft für Mathematik und Datenverarbeitung, 5205 St. Augustin 1, West Germany

Communicated by P. L. Butzer

DEDICATED TO MY ESTEEMED FATHER ON HIS SIXTY-FIFTH BIRTHDAY

The approximation of differentiable functions by algebraic polynomials Π_n with respect to the norm $\|f\|_F = \max_{i=1,\ldots,p} \|f^{(k_i)}\|$, where $0 = k_1 < k_2 < \cdots < k_p$ and $\|\cdot\|$ is the Chebyshev norm on $[a, b]$, is considered. The main result is the precise determination of the set of polynomials of best approximation in the norm $\|\cdot\|_F$. This is achieved by characterizing q, the smallest of the numbers k_i for which $P^{(k_i)}$ is the unique best approximation to $f^{(q)}$ from Π_{n-q} with respect to a norm $\|\cdot\|_G$ of the same type as above (see below). The dimension in question is precisely q.

Approximation in the norm $\|\cdot\|_F$ is nothing but the simultaneous approximation of a real valued function f and its derivatives $f^{(k_i)}$ in the Chebyshev norm by P and $P^{(k_i)}$. The simplest special case $p = 1$, $k_1 = 1$ of the problem was treated in detail by Moursund [8], who showed that a polynomial P of best approximation from Π_n to some function f is either unique, or else, P' is the unique polynomial of best approximation from Π_{n-1} to f' with respect to the Chebyshev norm. Our ma n theorem also contains a result of Johnson [5] who showed that if P and Q are polynomials of best approximation to some function f with respect to the seminorm $\max_i\{\| g^{(k_i)} \|\}$ where $0 \leqslant k_1 < k_2 < \cdots < k_p$, then $P^{(k_p)} = Q^{(k_p)}$. Other proofs of Moursund's result and of Johnson's result for the case $k_i = i - 1$ as well as characterizations of Kolmogorov type can be found in [1–4].

The main tools used in this paper are the notion of minimal polynomials of best approximation and theorems about Birkhoff interpolation.

Let $F = \{k_0, k_1, \ldots, k_p\}$ where $0 = k_0 < k_1 < \cdots < k_p$ are integers and $[a, b]$ be an interval. We introduce the Banach space $B_F = B$ of k_p-times

Copyright © 1975 by Academic Press, Inc.
All rights of reproduction in any form reserved.

continuously differentiable real-valued functions on $[a, b]$ with the norm $\|\cdot\|_F$ defined by

$$\|f\|_F = \max_{k_i \in F} \{\|f^{(k_i)}\|\},$$

where $\|\cdot\|$ is the Chebyshev norm.

Let $n \geqslant k_p$. We want to approximate $f \in B$ by an algebraic polynomial $P \in \Pi_n$ of degree not exceeding n with respect to the norm $\|\cdot\|_F$. We will denote this kind of approximation by "simultaneous approximation." That there exist best approximations follows easily from compactness arguments. We want to show that either there exists exactly one polynomial of best approximation or, if not, we want to characterize precisely all the polynomials of best approximation.

We begin with some terminology. Let $\Omega(f)$ be the set of best approximations from Π_n to f in B and set

$$\Lambda_{n,F}(f) = \inf_{P \in \Pi_n} \{\|f - P\|_F\},$$

which we will always assume to be nonzero. We define

$$\Omega(f) = \{P \mid P \in \Pi_n, \|f - P\|_F = \Lambda_{n,F}(f)\}.$$

We also define the extremal sets

$$U_i(P) = U_i(P, f) = \{x \in [a, b] \mid |f^{(k_i)}(x) - P^{(k_i)}(x)| = \Lambda_{n,F}(f)\},$$

which will play the same role as the set of extremal points in ordinary Chebyshev approximation. Clearly the U_i are compact.

We shall make use of "minimal" polynomials which are the algebraic interior points of the convex set $\Omega(f)$.

LEMMA 1. *There exists a* $\bar{P} \in \Omega(f)$ *(called a "minimal" polynomial) with the property that*

$$U_i(\bar{P}, f) \subset U_i(P, f)$$

and $\qquad i = 0, 1, \ldots, p,$

$$\bar{P}(x) = P(x), \qquad x \in U_i(\bar{P}, f),$$

for any other $f \in \Omega(f)$.

The proof can be carried out as in [6] or [7] or by use of the theory of convex sets. We note that the $U_i(\bar{P}, f)$ are independent of the choice of $\bar{P}$ and so we will just write $U_i(f)$ for them. It is clear that at least one of them is nonempty.

The following lemma is a modification of a characterization theorem of Kolmogorov type which is given in [8].

LEMMA 2. *$P \in \Omega(f)$ is a minimal polynomial of best approximation to $f \in B$ if and only if there exists no $Q \in \Pi_n$ with*

$$\max_{i=0,1,\ldots,p} \sup_{x \in U_i(f)} \{f^{(k_i)}(x) - P^{(k_i)}(x)\} Q^{(k_i)}(x) \leqslant 0$$

and such that the strict inequality

$$\{f^{(k_{i_0})}(x_{i_0}) - P^{(k_{i_0})}(x_{i_0})\} Q^{(k_{i_0})}(x_{i_0}) < 0,$$

holds for some i_0, $0 \leqslant i_0 \leqslant p$ and for some $x_{i_0} \in U_{i_0}(f)$.

Proof. By means of the usual continuity arguments, one can show that if $Q \in \Pi_n$ satisfies both of the above inequalities, then for all $\lambda > 0$ sufficiently small, $P_\lambda = P + \lambda Q$ is a polynomial of best approximation to f for which $U_i(P_\lambda, f) \subset U_i(P, f)$ for all i and that $x_{i_0} \notin U_{i_0}(P_\lambda, f)$. It follows that P cannot be a minimal polynomial of best approximation. Conversely, if P is not a minimal polynomial, then the difference $P - P_m$ of P and a minimal polynomial P_m satisfies the above inequalities for some k_{i_0} and x_{i_0}.

We need some more facts about the $U_i(P, f)$ for $P \in \Omega(f)$. If $f^{(k_i+1)}$ exists, $P \in \Omega(f)$ and if $x \in U_i(P, f)$, $x \neq a, b$, then

$$P^{(k_i+1)}(x) = f^{(k_i+1)}(x). \tag{1}$$

It is easy to see that if g and h are two continuous functions on $[a, b]$, differentiable at x, $a < x < b$, if $g \leqslant h$ in some neighborhood of x and if $g(x) = h(x)$, then $g'(x) = h'(x)$. Since $f^{(k_i+1)}$ exists, our statement follows immediately from this observation. Next, let $k_{i+1} = k_i + 1$ for some i. Then, for $P \in \Omega(f)$,

$$U_i(P, f) \cap U_{i+1}(P, f) \cap (a, b) = \varnothing. \tag{2}$$

Indeed, if $x \in U_i(P, f) \cap (a, b)$, then we have (1). That is

$$P^{(k_{i+1})}(x) = f^{(k_{i+1})}(x).$$

If also $x \in U_{i+1}$, then

$$|f^{(k_{i+1})}(x) - P^{(k_{i+1})}(x)| = \Lambda_{n,F} \neq 0,$$

which is not compatible with the last equality above.

For a function $f \in B_F$, the sets $U_i(\bar{P}, f)$ do not depend on the particular choice of the minimal polynomial $\bar{P}$ (by Lemma 1) and we write $U_i(f)$ for them. We put

$$G = \{k_i \mid k_i \in F, U_i(f) \neq \varnothing\} \tag{2}$$

and

$$q = \min_{k_i \in G} \{k_i\}. \tag{3}$$

By $G - q$, we mean the set $\{k_i - q \mid k_i \in G\}$. Thus $0 \in G - q$.

On the way to our main theorem, we give some results of interest in themselves. For a given $f \in B_F$, we consider the minimum

$$\min_{Q \in \Pi_n} \{\max_{k_i \in G} \| f^{(k_i)} - Q^{(k_i)} \|\} = \Lambda. \tag{3a}$$

THEOREM 3. *The number* (3a) *is equal to* $\Lambda(f)$ *and is achieved by each* $Q \in \Omega(f)$.

Proof. Clearly, $\Lambda \leqslant \Lambda(f)$. It remains to show that the maximum in (3a) is at least $\Lambda(f)$ for each $Q \in \Omega(f)$. Otherwise

$$\| f^{(k_i)} - Q^{(k_i)} \| \leqslant \Lambda(f) - \delta, \; k_i \in G,$$

for some $Q \in \Omega(f)$ and some $\delta > 0$. Let $M > 0$ be so large that $\| f^{(k_i)} \|$, $\| Q^{(k_i)} \| \leqslant M$ for all $k_i \in F$. If $\bar{P}$ is a minimal polynomial, there exists an $\epsilon > 0$ for which,

$$\| f^{(k_i)} - \bar{P}^{(k_i)} \| = \Lambda(f), \; k_i \in G,$$
$$\| f^{(k_i)} - P^{(k_i)} \| \leqslant \Lambda(f) - \epsilon, \; k_i \in F \backslash G.$$

Let $0 < \lambda < \frac{1}{2}$ be so small that $\lambda < \epsilon(4M)^{-1}$. We estimate the degree of approximation of the function f by the polynomial $Q_1 = \lambda Q + (1 - \lambda)\bar{P}$.

If $k_i \in G$, then

$$\| f^{(k_i)} - Q_1^{(k_i)} \| \leqslant \lambda \| f^{(k_i)} - Q^{(k_i)} \| + (1 - \lambda) \| f^{(k_i)} - \bar{P}^{(k_i)} \|$$
$$< \lambda(\Lambda(f) - \delta) + (1 - \lambda)\Lambda(f) = \Lambda(f) - \lambda\delta < \Lambda(f).$$

On the other hand, if $k_i \in F \backslash G$,

$$\| f^{(k_i)} - Q_1^{(k_i)} \| < 2M\lambda + (1 - \lambda)(\Lambda(f) - \epsilon)$$
$$< \epsilon/2 + \Lambda(f) - (1 - \lambda)\epsilon < \Lambda(f).$$

Thus, $\| f - Q_1 \|_F < \Lambda(f)$, a contradiction which completes the proof.

The following theorem gives a sufficient condition under which the approximation of f in the norm $\|\cdot\|_F$ is unique.

THEOREM 4. *If for the function f, which is $k_p + 1$-times differentiable, we have $q = 0$, that is $U_0(f) \neq \varnothing$, then f has a unique polynomial of best approximation in the norm $\|\cdot\|_F$.*

Remark. The more natural assumption that $f \in B_F$ cannot be used to replace the existence of the $k_p + 1$-st derivative of f. This can be seen by the counterexample which is given in Chalmers [3]. It is an example of a function which is differentiable but has no second derivative and which has more than one best approximation by quadratic polynomials with respect to the norm $\|f\|_s = \max\{\|f\|, \|f'\|\}$ even though $U_0(f) \neq \varnothing$. Although there is a small mistake in Chalmer's example, his claim holds none the less.

From the proof of Theorem 4, one can however see that if $U_p(f) = \varnothing$, then the assumptions $f \in B_F$ and $U_0(f) \neq \varnothing$ suffice for the uniqueness of the best approximation.

Proof of Theorem 4. Let P be an element of $\Omega(f)$, and let $\bar{P}$ be a minimal polynomial with $P = \bar{P}$. From the definition of a minimal polynomial, we have $P^{(k_i)}(x) = \bar{P}^{(k_i)}(x)$, $x \in U_i$, $i = 0, 1,..., p$. From (1), we get

$$\bar{P}^{(k_i+1)}(x) = f^{(k_i+1)}(x) = P^{(k_i+1)}(x), \quad x \in U_i(f) \cap (a, b), \quad i = 0,..., p.$$

Thus it is sufficient to show that only the polynomial R which is identically zero satisfies the conditions

$$R^{(k_i)}(x) = 0, \qquad x \in U_i(f), \qquad i = 0, 1,..., p, \tag{4}$$

$$R^{(k_{i+1})}(x) = 0, \qquad x \in U_i(f) \cap (a, b). \tag{5}$$

This is done by means of known theorems about the Birkhoff interpolation problem. In the following, we use the terms, notations and results given in G. G. Lorentz [11].

The sets $U_i = U_i(f)$ might be infinite. For each i for which this happens, we omit some of the elements $x \in U_i$ leaving arbitrary $n + 2$ of them. We still denote the new sets by U_i. By l_i, we denote the number of points of U_i, by e_i, the number of points of U_i which are a or b. Let E be the incidence matrix corresponding to the Birkhoff interpolation problem (4) and (5).

LEMMA 5. *The matrix E of problem* (4), (5) *has*

$$N + 1 = \sum_{i=0}^{p} (2l_i - e_i) \tag{6}$$

entries; it is a free matrix for the Birkhoff interpolation problem for polynomials of degree N. Moreover, $N \geqslant n+1$.

Proof. First of all, we note that the conditions (4) and (5) cannot overlap. This follows from relation (2). Hence the number of different equations (4) and (5) is equal to the sum of the number of points in all sets U_i, $U_i \cap (a, b)$. That is, it is given by (6).

By a theorem of Atkinson and Sharma [10, 11], an incidence matrix is free it it has no odd supported sequences and satisfies the strong Pólya condition. However, all conditions (4) and (5) involving points from the interior of $[a, b]$ come in nonoverlapping pairs. This means that any sequence of ones in E not lying in the first or the last row of E is even.

Thus it remains to check that E satisfies the strong Pólya condition. This condition can be written

$$\sum_{s=0}^{k} m_s \geqslant k+2, \qquad 0 \leqslant k \leqslant N, \tag{7}$$

where m_s is the number of ones in the sth column of E, or, what is the same, the number of Eqs. (4) for which $k_i \leqslant s$ and of Eqs. (5) for which $k_i+1 \leqslant s$.

We first note that (7) is satisfied for $k=0$, which means that $m_0 \geqslant 2$. By assumption, $U_0 \neq \varnothing$ and so $\|f-\bar{P}\| = A(f)$. Since $\bar{P}+c$, where c is a constant, has the same derivatives as $\bar{P}$, $\bar{P}+c$ cannot be a better Chebyshev approximation to f than $\bar{P}$ is, because then $\bar{P}+c$ would have smaller extremal sets than $\bar{P}$. It follows from this observation that $|f(x)-\bar{P}(x)|$ attains its maximum at at least two points and so $m_0 \geqslant 2$.

Let (7) be violated for some k and let $\bar{k}$ be the smallest such k. Then $\bar{k} \leqslant n$ and $\bar{k} \geqslant 1$. Moreover, $\sum_{s=0}^{\bar{k}+1} m_s \geqslant \bar{k}+1$, but $\sum_{s=0}^{\bar{k}} m_s \leqslant \bar{k}+1$. This implies that $\sum_{s=0}^{\bar{k}} m_s = \bar{k}+1$ and that $m_{\bar{k}} = 0$.

Let $\bar{E}$ denote the incidence matrix consisting of the first $\bar{k}+1$ columns of E. Like E, the matrix $\bar{E}$ has no odd supported sequences. Moreover, from (7), $\bar{E}$ satisfies the strong Pólya conditions. Then $\bar{E}$ is free for polynomials of degree not exceeding $\bar{k}$. Therefore, one can find a polynomial R, of degree at most $\bar{k}$, which satisfies the conditions

$$R^{(k_s)}(x) = -\sigma[f^{(k_s)}(x) - \bar{P}^{(k_s)}(x)],\ x \in U_i,\ k_s \leqslant \bar{k}, \tag{8}$$

$$R^{(k_s+1)}(x) = 0,\ x \in U_i \cap (a, b),\ k_s+1 \leqslant \bar{k}. \tag{9}$$

Here $\sigma(\alpha)$ denotes the sign of α. In addition,

$$R^{(k_s)}(x) = 0,\ k_s > \bar{k}. \tag{10}$$

This contradicts Lemma 2 and proves Lemma 5.

From Lemma 3, it follows immediately that $P \in \Omega(f)$ is unique.

MAIN THEOREM. *Let f be $k_p + 1$-times differentiable and let G and q be defined by* (2) *and* (3). *Then $\Omega(f)$ is a convex set of dimension q. In addition, for $P \in \Omega(f)$, the derivative $P^{(q)}$ is unique.*

Proof. The condition $\|f - P\|_F \leq \Lambda(f)$ is equivalent to the two conditions

$$\|f^{(k_s)} - P^{(k_s)}\| \leq \Lambda(f),\ k_s < q, \tag{11}$$

$$\|f^{(k_s)} - P^{(k_s)}\| \leq \Lambda(f),\ k_s \geq q. \tag{12}$$

From Theorem 3, we see that (12) means exactly that $P^{(q)}$ is the polynomial of best approximation in the norm $\|\cdot\|_{G-q}$ to $f^{(q)}$. This defines $P^{(q)}$ uniquely.

Then also P is defined uniquely up to a polynomial of degree not exceeding q, so that $\dim \Omega(f) \leq q$. On the other hand, let $\bar{P}$ be a minimal polynomial in $\Omega(f)$. Then conditions (12) are satisfied, while instead of (11) we have

$$\|f^{(k_s)} - \bar{P}^{(k_s)}\| < \Lambda(f),\ k_s < q.$$

If Q is an arbitrary polynomial of degree not exceeding q with sufficiently small coefficients, both (11) and (12) will be satisfied for $P = \bar{P} + Q$. All these polynomials belong to $\Omega(f)$.

REFERENCES

1. E. BREDENDIEK, Simultanapproximation, Ph.D. Dissertation, Hamburg, 1968.
2. E. BREDENDIEK, Simultanapproximation, *Arch. Rat. Math. Anal.* **33** (1969), 307–330.
3. B. CHALMERS, Uniqueness of best approximation of a function and its derivatives, *J. Approximation Theory* **7** (1973), 213–225.
4. M. HOLLENHORST, Die Approximation von Funktionen und ihren Ableitung, Masters thesis, Bonn, 1968.
5. L. JOHNSON, Unicity in approximation of a function and its derivatives, *Math. Comp.* **22** (1968), 873–875.
6. G. G. LORENTZ AND K. L. ZELLER, Monotone approximation by algebraic polynomials, *Trans. Amer. Math. Soc.* **149** (1970), 1–17.
7. R. A. LORENTZ, Uniqueness of best approximation by monotone polynomials, *J. Approximation Theory* **4** (1971), 401–418.
8. D. G. MOURSUND, Chebyshev approximation of a function and its derivatives, *Math. Comp.* **18** (1964), 382–389.
9. I. J. SCHOENBERG, On Hermite–Birkhoff interpolation, *J. Math. Anal. Appl.* **16** (1966), 538–543.
10. K. ATKINSON AND A. SHARMA, A partial characterization of poised Hermite–Birkhoff interpolation problems, *SIAM J. Numer. Anal.* **6** (1969), 230–235.
11. G. G. LORENTZ, G. D. Birkhoff's theory, preprint.